On the geometric quantization of contact manifolds

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Abstract

Suppose that (M, E) is a compact contact manifold, and that a compact Lie group G acts on M transverse to the contact distribution E. In an earlier paper, we defined a G-transversally elliptic Dirac operator $\rlap{/}{D}_b$, constructed using a Hermitian metric h and connection ∇ on the symplectic vector bundle $E \to M$, whose equivariant index is well-defined as a generalized function on G, and gave a formula for its index. By analogy with the geometric quantization of symplectic manifolds, the $\mathbb{Z}_2$-graded Hilbert space $\ker \rlap{/}{D}_b \oplus \ker \rlap{/}{D}_b^*$ can be interpreted as the "quantization" of the contact manifold (M, E); the character of the corresponding virtual G-representation is then given by the equivariant index of $\rlap{/}{D}_b$. By defining contact analogues of the algebra of observables, pre-quantum line bundle and polarization, we further extend the analogy by giving a contact version of the Souriau-Kostant approach to quantization.

1   Introduction

The study of geometric quantization dates back to the work of Souriau [1] and Kostant [2] in the 1960s. The standard setting is that of symplectic geometry: viewing a symplectic manifold (M, ω) as a mathematical model for a classical Hamiltonian system, we may ask whether or not one can associate to (M, ω) some Hilbert space representing a corresponding quantum mechanical system. The basic principles that we expect such a correspondence to obey include the following: classical observables should be associated to quantum ones, and if the classical system has symmetries, given by a Hamiltonian action of a Lie group G on (M, ω), then there should be a representation of the same group of symmetries on the corresponding quantum Hilbert space.

We note that there are a number of different mathematical frameworks that fall under the title of 'quantization,' including the theory of deformation quantization in the related field of Poisson geometry [3]. Quantization in the case of contact geometry, which is also closely related to symplectic geometry, seems to have been less studied. The deformation quantization of a Poisson structure near a contact boundary is studied in [4], and an approach using D-modules appears in [5]. Other related results include the quantization of Jacobi manifolds in [6], and the papers of Guillemin and Sternberg on homogeneous quantization,

beginning with [7]. However, to our knowledge, there is no explicit geometric quantization procedure for contact manifolds that includes an analogue coming from the point of view of the equivariant index theorem.

The motivation for this note was the observation of certain similarities between results appearing in the author's thesis [8], and in [9] for the case of contact manifolds, and the formulation of geometric quantization in terms of the equivariant index theorem. (See [10] for example, or [11] for an outline of this approach.) These similarities suggested that there may exist an procedure of geometric quantization for contact manifolds analogous to the symplectic case, for which the equivariant index formula introduced in [9] provides a reformulation in terms of index theory.

We will begin with a quick review of the standard material on geometric quantization, following which we will proceed to give contact analogues of each of the ingredients in the symplectic case. In particular, we describe the algebra of observables, the contact momentum map associated to a group of symmetries, and the corresponding subalgebra of observables. We also will introduce an algebra $\mathcal{P}_b(M, \Omega)$ that could be called the "Poisson algebra of the contact distribution," and show that this algebra is isomorphic to the Lie algebra of infinitesimal symmetries of (M, θ). When our contact manifold is a circle bundle over a symplectic manifold, this algebra corresponds to the pullback to M of the Poisson algebra of the symplectic manifold. Moreover, the symplectic manifold in this case will be prequantizable, and the contact manifold is a prequantum circle bundle. In this setting we note a close connection to the Kirillov-Kostant approach to equivariant prequantization.

We then give the construction of the Hilbert space of sections of a line bundle analogous to the prequantum line bundle, and describe an analogue of polarization in terms of CR structures. The main result of this section is that a natural candidate for the quantization of a contact manifold with compatible CR structure is the space of CR-holomorphic functions on the manifold.

We then review the description of geometric quantization as an equivariant index, and give an analogous formulation in contact geometry. We describe contact analogues of Clifford and spinor modules, Clifford connections, and Dirac operators. In the case of a CR-holomorphic vector bundle $(\mathcal{W}, \overline{\partial}_{\mathcal{W}}) \to (M, E_{1,0})$, we will show that our Dirac operator is given by

$$\mathbb{D}_b = \sqrt{2} \left(\overline{\partial}_{\mathcal{W}} + \overline{\partial}_{\mathcal{W}}^* \right).$$

As a result, the index of $\mathbb{D}_b$ can be expressed in terms of the cohomology of the $\overline{\partial}_{\mathcal{W}}$ operator; in particular, when $\mathcal{W} = M \times \mathbb{C}$, we can relate this index to the Kohn-Rossi cohomology groups [12].

Finally, we make use of the results from [9, 8] to show that although the quantization $Q(M)$ of a contact manifold (M, E) obtained in this way is infinite-dimensional, if a compact Lie group G acts on M such that the group orbits are nowhere tangent to the contact distribution E, then, by a result of Atiyah [13], the character of the corresponding G-representation on $Q(M)$ is defined as a generalized function on G, and given by an equivariant index formula similar to the equivariant Riemann-Roch formula in the symplectic case.

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2 Geometric quantization of symplectic manifolds

Since the material presented here is quite standard, we will try to be brief, and refer the reader to the texts [3, 14, 15] for details. Let (M, ω) be a compact symplectic manifold. The classical dynamics are given by Hamilton's equations: to any Hamiltonian function $H \in C^\infty(M)$ we can associate the unique vector field $X_H \in \mathfrak{X}(M)$ satisfying

$$dH = \iota(X_H)\omega.$$

Such vector fields are *symplectic*, in the sense that the flow of X_H preserves the symplectic form ω. Moreover, the integral curves of X_H lie in level sets of H. The algebra of observables is the Poisson algebra $C^\infty(M)$, equipped with the Poisson bracket $\{f, g\} = \omega(X_f, X_g)$. We now suppose that a compact Lie group G acts on M, preserving ω. This gives us a map

$$\mathfrak{g} \to \mathfrak{X}_{symp}(M)$$
$$X \mapsto X_M,$$

where $\mathfrak{X}_{symp}(M)$ denotes the space of symplectic vector fields, and X_M is the vector field generated by the infinitesimal action of $\mathfrak{g}$ on M. The action of G is called *Hamiltonian* if this map factors through the map $C^\infty(M) \to \mathfrak{X}_{symp}(M)$ given by associating a function to its Hamiltonian vector field.

Definition 2.1. *A **momentum map** is an equivariant map $\Phi : M \to \mathfrak{g}^*$ such that for each $X \in \mathfrak{X}$, the pairing $\Phi^X = < \Phi, X >$ satisfies*

$$d\Phi^X = \iota(X_M)\omega. \tag{1}$$

Such a momentum map exists if and only if the action of G on (M, ω) is Hamiltonian [16, 17]). The desired mapping $\mathfrak{g} \to C^\infty(M)$ is given by $X \mapsto \Phi^X$. We note that this map is a Lie algebra homomorphism with respect to the Lie algebra structure on $C^\infty(M)$ given by the Poisson bracket. The functions $\{\Phi^X | X \in \mathfrak{g}\}$ thus generate a Lie subalgebra of $C^\infty(M)$.

To our symplectic manifold (M, ω) we wish to associate a Hilbert space $\mathcal{H}$, such that the action of G on M corresponds to a representation of G on $\mathcal{H}$. Moreover, classical 'observables' should correspond to quantum ones: there should be an algebra of skew-Hermitian operators A_X on $\mathcal{H}$ and Lie algebra isomorphism between this algebra (with respect to the commutator bracket) and the algebra generated by the momentum map components Φ^X (with respect to the Poisson bracket).

Suppose we are given a Hamiltonian G-space (M, ω, Φ), such that the equivariant cohomology class of $\omega(X) = \omega - \Phi(X)$ is integral. Then there exists a G-equivariant complex line bundle $\pi : \mathbb{L} \to M$, equipped with G-invariant Hermitian metric h and connection ∇ with equivariant curvature $F_\nabla(X) = \pi^*\omega(X)$. Such a line bundle $\mathbb{L}$ is known as a G-**equivariant prequantum line bundle** for (M, ω, Φ). The action of G on M induces a linear action of G on the space of sections of $\mathbb{L}$ by bundle automorphisms, and obtain a unitary representation of G on the Hilbert space

$$\mathcal{H} = \Gamma_{L^2}(M, \mathbb{L})$$

of L^2 sections of $\mathbb{L}$, with respect to the inner product

$$< s_1, s_2 > = \int_M h(s_1, s_2) \frac{\omega^n}{n!}.$$

From the infinitesimal action of $\mathfrak{g}$ on the space of sections, we obtain the desired correspondence $\Phi^X \mapsto A_X$ between classical and quantum observables via

$$A_X = -\nabla_{X_M} + i\pi^* \Phi^X.$$

The Hilbert space we obtain in this way turns out to be too big (for example, in the non-compact case $M = T^*X$, we obtain $L^2(T^*X)$ rather than $L^2(X)$). The standard way of cutting down the space of sections is to apply a *polarization*. We will restrict ourselves to the case of a complex polarization, which is defined to be an integrable Lagrangian subbundle $\mathcal{P}$ of $TM \otimes \mathbb{C}$ such that $\mathcal{P} \cap \overline{\mathcal{P}} = 0$.

In other words, a polarization is given by a complex structure on M that is compatible with the symplectic structure. The existence of a complex polarization is thus equivalent to having a Kähler structure on M. A polarization determines a subspace of the space of L^2 sections of $\mathbb{L}$ by requiring $\nabla_{\overline{X}} s = 0$ for all $X \in \mathcal{P}$; these are the so-called *polarized* sections. The space of polarized sections is then a candidate for the space $Q(M)$. By [14, Proposition 6.30], there is a unique holomorphic structure on $\mathbb{L}$ such that the (local) polarized sections of $\mathbb{L}$ are the (local) holomorphic sections of $\mathbb{L}$. That is, the connection ∇ preserves the metric, and satisfies $\nabla^{0,1} = \overline{\partial}$, where $\nabla^{0,1} = \nabla|_{\overline{\mathcal{P}}}$.

Remark 2.2. We can see here why it is advantageous to work with sections of a complex line bundle, rather than complex functions on M: the space $Q(M) = \Gamma_{hol}(M, \mathbb{L})$ is non-trivial if $\mathbb{L}$ is sufficiently positive, while this is not the case for the space of holomorphic functions on the compact manifold M. Moreover, we can generalize slightly by replacing the space of holomorphic sections of $\mathbb{L}$ by the sheaf cohomology groups $H^k(M; \mathcal{O}(\mathbb{L}))$. The definition of $Q(M)$ is then taken to be the $\mathbb{Z}_2$-graded vector space

$$Q(M) = \sum (-1)^k H^k(M; \mathcal{O}(\mathbb{L})). \tag{2}$$

The corresponding action of G on these groups is well-understood [18], and moreover, the groups $H^k(M; \mathcal{O}(\mathbb{L}))$ are isomorphic to the Dolbeault cohomology groups $H^{0,k}(M; \mathbb{L})$ of the complex of differential forms on M with values in $\mathbb{L}$. This observation will later allow us to think of $Q(M)$ in terms of equivariant index theorem.

3 Geometric quantization of contact manifolds

3.1 Contact momentum maps

Let (M, E) be a compact contact manifold of dimension $2n + 1$. We will assume that the contact distribution E is cooriented, so that there exists a global contact form $\theta \in \Gamma(E^0 \setminus 0)$. The contact form θ determines a splitting $T^*M = E^* \oplus E^0$ of the cotangent bundle, a trivialization $E^0 = M \times \mathbb{R}$, and an orientation on M given by the volume form $\mu = \theta \wedge \dfrac{d\theta^n}{n!}$.

We suppose that a compact Lie group G acts on M by contact transformations; by averaging, we may assume that the contact form θ is G-invariant.

Definition 3.1. *We define the* **contact momentum map** *associated to the contact form* θ *to be the map* $\Phi_\theta : M \to \mathfrak{g}^*$ *such that for any* $X \in \mathfrak{g}$*, we have*

$$< \Phi_\theta, X >= \theta(X_M). \tag{3}$$

Remark 3.2. The contact momentum map defined above does of course depend on the choice of contact form θ. For further discussion of the properties of contact momentum maps, see [19].

We note that the momentum map components $\Phi_\theta^X = \theta(X_M)$ satisfy similar properties to the components of a symplectic momentum map. In particular, Φ_θ^X is the 'Hamiltonian' function associated to the vector field X_M, in the sense that, by the invariance of θ, we have

$$d\Phi_\theta^X = d\iota(X_M)\theta = -\iota(X_M)d\theta = \iota(X_M)\Omega,$$

where $\Omega = -d\theta$ is the symplectic structure on the fibres of the contact distribution E determined by θ.

3.2   The Jacobi algebra

Given the action of G on (M, θ) leaving θ invariant, the vector fields X_M generated by the Lie algebra elements $X \in \mathfrak{g}$ are contact, in the sense that $\mathcal{L}(X_M)\theta = 0$ for all $X \in \mathfrak{g}$. (In general, a contact vector field V satisfies $\mathcal{L}(V)\theta = f\theta$ for some $f \in C^\infty(M)$.) To continue the analogy with symplectic geometry, we need a notion of Hamiltonian vector field, and a Lie algebra structure on $C^\infty(M)$. We refer to the article [20] for much of the material presented in this section.

Definition 3.3. *A* **Jacobi structure** *on a manifold M is a bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ that is skew-symmetric, satisfies the Jacobi identity, and is local, in the sense that the support of $\{f, g\}$ is contained in the intersection of the supports of f and g.*

A Jacobi structure is equivalent to the existence of a bivector field $\Lambda \in \Gamma(\bigwedge^2(TM))$ and a vector field $\xi \in \mathfrak{X}(M)$ such that

$$[\xi, \Lambda] = \mathcal{L}(\xi)\Lambda = 0 \quad \text{and} \quad [\Lambda, \Lambda] = 2\xi \wedge \Lambda,$$

where $[\cdot, \cdot]$ denotes the Schouten bracket. The relationship between the Jacobi bracket and the data (Λ, ξ) is given by

$$\{f, g\} = \Lambda(df, dg) + \iota(\xi)(f\, dg - g\, df).$$

Given a contact manifold (M, E) equipped with contact form θ, the vector field ξ is given by the *Reeb field*, defined to be the unique vector field such that

$$\iota(\xi)\theta = 1 \quad \text{and} \quad \iota(\xi)\Omega = 0,$$

where $\Omega = -d\theta$. The contact form also determines a map $\Lambda^\# : T^*M \to TM$ such that, for any $\eta \in T^*M$, we have

$$\theta(\Lambda^\#(\eta)) = 0 \quad \text{and} \quad \iota(\Lambda^\#(\eta))\Omega = \eta - (\eta(\xi))\theta.$$

We note that the image of the map $\Lambda^{\#}$ is contained in the contact distribution E, by the first of the above two conditions. Finally, we can define $\Lambda \in \Gamma(\bigwedge^2(TM))$ by

$$\Lambda(\eta, \zeta) = \iota(\Lambda^{\#}(\eta))\zeta = -\iota(\Lambda^{\#}(\zeta))\eta.$$

From the Jacobi structure associated to the contact form θ, we obtain a Lie algebra structure on $C^{\infty}(M)$, as well as a notion of Hamiltonian vector field:

Definition 3.4. *For any $f \in C^{\infty}(M)$, the **Hamiltonian vector field** associated to f is the vector field*

$$X_f = \Lambda^{\#}(df) + f\xi. \tag{4}$$

Proposition 3.5. *For any $f \in C^{\infty}(M)$, the associated Hamiltonian vector field X_f satisfies the following properties:*

1. *The map $f \mapsto X_f$ is a Lie algebra homomorphism: for any $f, g \in C^{\infty}(M)$, we have*

$$X_{\{f,g\}} = [X_f, X_g].$$

2. *For any $g \in C^{\infty}(M)$, $X_f \cdot g = \{f, g\} + (\xi \cdot f)g$.*

3. *$\iota(X_f)\Omega = df - (\xi \cdot f)\theta$.*

4. *X_f is a contact vector field: $\mathcal{L}(X_f)\theta = (\xi \cdot f)\theta$.*

Proof. The first property is well-known and appears for example in [6] and [20]. The proof of each of the remaining properties is obtained by a straight-forward calculation. For the second, we have

$$X_f \cdot g = \iota(\Lambda^{\#}(df) + f\xi)dg = \Lambda(df, dg) + f\xi \cdot g = \{f, g\} + g\xi \cdot f.$$

For the third, we have

$$\iota(X_f)\Omega = \iota(\Lambda^{\#}(df))\Omega + f\iota(\xi)\Omega = df - (\iota(\xi)df)\theta,$$

and the fourth now follows from the third by Cartan's formula, since $\iota(X_f)d\theta = -\iota(X_f)\Omega$, and $d\iota(X_f)\theta = df$. $\qquad\square$

3.3 The Poisson algebra

From the above proposition, we see that the image of the homomorphism $C^{\infty}(M) \to \mathfrak{X}(M)$ given by (4) is contained in the Lie subalgebra of contact vector fields. Moreover, we note that in each case, the failure of $(C^{\infty}(M), \{\cdot, \cdot\})$ to behave like a Poisson algebra is indicated by the presence of the term $\xi \cdot f$. We therefore might ask what can be said about those functions for which $\xi \cdot f = 0$. For any manifold M equipped with a closed two-form Ω, we have the associated Poisson algebra [14]

$$\mathcal{P}(M, \Omega) = \{(f, X) \in C^{\infty}(M) \times \mathfrak{X}(M) | df = \iota(X)\Omega\}.$$

The bracket is given by $[(f, X), (g, Y)] = (\frac{1}{2}(Y \cdot f - X \cdot g), [X, Y])$, and the Poisson algebra acts on M via $(f, X) \mapsto X$. Of course, if Ω is symplectic, then $\mathcal{P}(M, \Omega)$ is isomorphic to $C^\infty(M)$, since each f is associated to a unique Hamiltonian vector field X_f.

Let us suppose instead that (M, θ) is a contact manifold, and consider the Poisson algebra $\mathcal{P}(M, \Omega)$ with respect to the two-form $\Omega = -d\theta$. For any $f \in C^\infty(M)$, we can consider the pair (f, X_f), where $X_f = \Lambda^\#(df) + f\xi$, as above. By Proposition 3.5, we see that $\iota(X_f)\Omega = df$ if and only if $\xi \cdot f = 0$.

Definition 3.6. *We denote by $\mathcal{P}_b(M, \Omega)$ the subset of $\mathcal{P}(M, \Omega)$ given by*

$$\mathcal{P}_b(M, \Omega) = \{(f, X_f) \in C^\infty(M) \times \mathfrak{X}(M) | \xi \cdot f = 0\}.$$

We note that $\mathcal{P}_b(M, \Omega)$ is a proper subset of $\mathcal{P}(M, \Omega)$, since $(f, X_f + g\xi) \in \mathcal{P}(M, \Omega)$ for any $g \in C^\infty(M)$. However, we have the following:

Proposition 3.7. *The set $\mathcal{P}_b(M, \Omega)$ is a Lie subalgebra of $\mathcal{P}(M, \Omega)$.*

Proof. For any $f, g \in C^\infty(M)$, we see using Proposition 3.5 that

$$[(f, X_f), (g, X_g)] = (\{f, g\} + \frac{1}{2}(g\xi \cdot f - f\xi \cdot g), [X_f, X_g]).$$

Since $[X_f, X_g] = X_{\{f,g\}}$, when $\xi \cdot f = \xi \cdot g = 0$, we have $[(f, X_f), (g, X_g)] = (\{f, g\}, X_{\{f,g\}})$. $\square$

Using the above, and the fact that $\xi \cdot f = 0$ if and only if $\iota(X_f)\Omega = df$, we obtain the following:

Proposition 3.8. *If we define the subsets $C_b^\infty(M) = \{f \in C^\infty(M) | \xi \cdot f = 0\}$ and $\mathfrak{X}_b(M) = \{X_f \in \mathfrak{X}_{ham}(M) | \iota(X_f)\Omega = df\}$, then we have:*

1. *The space $C_b^\infty(M)$ is a Lie subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$.*

2. *The space $\mathfrak{X}_b(M)$ is a Lie subalgebra of $(\mathfrak{X}(M), [\cdot, \cdot])$.*

3. *We have Lie algebra isomorphisms $\mathcal{P}_b(M, \Omega) \cong C_b^\infty(M) \cong \mathfrak{X}_b(M)$.*

Let us further denote by $\mathfrak{X}_{symm}(M, \theta) = \{X \in \mathfrak{X}(M) | \mathcal{L}(X)\theta = [X, \xi] = 0\}$ the Lie algebra of infinitesimal symmetries of (M, θ). Following [14], we have:

Proposition 3.9. *The map $\mathcal{P}_b(M, \Omega) \to \mathfrak{X}_{symm}(M, \theta)$ given by $(f, X_f) \mapsto X_f$ is an isomorphism of Lie algebras.*

Proof. By Proposition 3.5, we see that the action of the pair $(f, X_f) \in \mathcal{P}(M, \Omega)$ on M preserves the contact form, since $\mathcal{L}(X_f)\theta = (\xi \cdot f)\theta = 0$. Moreover, we have

$$[X_f, \xi] = [\Lambda^\#(df), \xi] + f[\xi, \xi] = 0,$$

so that X_f is an infinitesimal symmetry of (M, θ). Conversely, choose any $X \in \mathfrak{X}_{symm}(M, \theta)$. Since $[X, \xi] = 0$, it follows that $X = Y + f\xi$, where $Y \in E = \ker \theta$. Since $\mathcal{L}(X)\theta = 0$, we have

$$0 = \mathcal{L}(Y + f\xi)\theta = \iota(Y + f\xi)d\theta + d\iota(Y + f\xi)\theta = \iota(Y)d\theta + df,$$

and therefore, $df = -\iota(Y)d\theta = \iota(Y)\Omega$. Since $\iota(Y)\Omega = df$ and $\iota(Y)\theta = 0$, it follows that $Y = \Lambda^\#(df)$, and thus, $X = X_f$. $\square$

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Let us now consider the case where (M, θ) is a Boothby-Wang fibration [21]. That is, (M, θ) is a principal $U(1)$-bundle over a symplectic manifold (B, ω), with connection 1-form θ (identifying $\mathfrak{u}(1)$ with $\mathbb{R}$). The symplectic manifold (B, ω) is then prequantizable, and $\mathbb{L} = M \times_{U(1)} \mathbb{C}$ is the associated prequantum line bundle. Given an action of G on (M, θ) preserving θ, the prequantization condition becomes $\pi^*\omega = -d\theta$ and $\pi^*\Phi^X = \iota(X_M)\theta$.

As outlined in [14], the traditional (Kirillov-Kostant) approach is to start with a Hamiltonian action of G on (B, ω), and lift the infinitesimal action to M such that $\pi^*\Phi^X = \theta(X_M)$. However, one can in fact lift the action of the entire Poisson algebra $C^\infty(B)$ to M: given $f \in C^\infty(B)$, let X_f be its associated Hamiltonian vector field. The action of f on M is then given by

$$f \mapsto X_f^{hor} + \pi^*f \cdot \xi, \tag{5}$$

where X_f^{hor} denotes the horizontal lift of X_f with respect to the connection θ, and the Reeb field ξ is the infinitesimal generator of the $U(1)$ action. By [14, Proposition 6.17], the Poisson algebra of (B, ω) is isomorphic via the above map to $\mathfrak{X}_{symm}(M)$, and thus, $\mathcal{P}(B, \omega)$ is isomorphic to $\mathcal{P}_b(M, \Omega)$. Moreover, we see that the vector field (5) is the Hamiltonian vector field (in the Jacobi sense, given by (4)) associated to π^*f.

Remark 3.10. One advantage of our approach is that the algebra $\mathcal{P}_b(M, \Omega)$ makes sense even when (M, θ) is not a regular contact manifold, and therefore can be applied in settings where no regular contact structure exists (such as the 3-torus [22]). Moreover, we notice that when the lifts of the momentum map components Φ^X satisfy the prequantization condition, they exactly coincide with the components of the contact momentum map.

In Section 3.6 below, we will see that the trivial line bundle $\mathbb{L} = M \times \mathbb{C}$ serves as a contact version of the prequantum line bundle. When (M, θ) is a prequantum circle bundle, we note that $\mathbb{L}/U(1) = M \times_{U(1)} \mathbb{C}$ is a prequantum line bundle for the symplectic manifold $M/U(1)$.

3.4 Contact momentum maps revisited

For any $X \in \mathfrak{g}$ we have the function $\Phi_\theta^X \in C^\infty(M)$ given in terms of the contact momentum map. Continuing the analogy with symplectic geometry, we have the following:

Proposition 3.11. *Suppose a compact Lie group G acts on a compact contact manifold preserving a chosen contact form θ, and let $\Phi_\theta : M \to \mathfrak{g}^*$ denote the corresponding contact momentum map. With respect to the Jacobi structure defined by θ, we have the following:*

1. *The map $\mathfrak{g} \to C^\infty(M)$ given by $X \to \Phi_\theta^X$ is a Lie algebra homomorphism.*

2. *The Hamiltonian vector field associated to Φ_θ^X is equal to X_M.*

Proof. Let $\Omega = -d\theta$, and note that $\iota(\xi)\Omega = 0$, and $\iota(X_M)\Omega = d\Phi_\theta^X$. The Jacobi bracket is

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given by

$$
\begin{aligned}
\{\Phi_\theta^X, \Phi_\theta^Y\} &= \Lambda(d\Phi_\theta^X, d\Phi_\theta^Y) + \iota(\xi)(\Phi_\theta^X \, d\Phi_\theta^Y - \Phi_\theta^Y \, d\Phi_\theta^X) \\
&= \Lambda(\iota(X_M)\Omega, \iota(Y_M)\Omega) + \Phi_\theta^X \Omega(Y_M, \xi) - \Phi_\theta^Y \Omega(X_M, \xi) \\
&= \Omega(Y_M, \Lambda^\#(\iota(X_M)\Omega)) \\
&= -\iota(Y_M)[\iota(\Lambda^\#(\iota(X_M)\Omega))\Omega] \\
&= -\iota(Y_M)(\iota(X_M)\Omega - \Omega(X_M, \xi), \theta) \\
&= \Omega(Y_M, X_M),
\end{aligned}
$$

while the component of Φ_θ in the direction of $[X, Y]$ is given by

$$
\begin{aligned}
\Phi_\theta^{[X,Y]} &= \iota([X,Y]_M)\theta = \iota([X_M, Y_M])\theta \\
&= [\mathcal{L}(X_M), \iota(Y_M)]\theta \\
&= \mathcal{L}(X_M)(\iota(Y_M)\theta) + \iota(Y_M)(\mathcal{L}(X_M)\theta) \\
&= \iota(X_M)d(\iota(Y_M)\theta) \\
&= -\iota(X_M)\iota(Y_M)d\theta = \Omega(Y_M, X_M),
\end{aligned}
$$

using the invariance of θ. This establishes the first point. For the second, we note that the Hamiltonian vector field associated to $f = \Phi_\theta^X$ is given by

$$
\begin{aligned}
X_f &= \Lambda^\#(d\Phi_\theta^X) + \Phi_\theta^X \xi \\
&= \Lambda^\#(\iota(X_M)\Omega) + \Phi_\theta^X \xi.
\end{aligned}
$$

We now compute $\iota(X_f)\theta$ and $\iota(X_f)\Omega$. We have

$$
\iota(X_f)\theta = \iota(\Lambda^\#(\iota(X_M)\Omega))\theta + \iota(\Phi_\theta^X \xi)\theta = \Phi_\theta^X = \iota(X_M)\theta,
$$

and

$$
\iota(X_f)\Omega = \iota(\Lambda^\#(\iota(X_M)\Omega))\Omega + \iota(\Phi_\theta^X \xi)\Omega = \iota(X_M)\Omega - \Omega(X_M, \xi)\theta = \iota(X_M)\Omega.
$$

$\square$

Thus, we see that any group action preserving the contact distribution is Hamiltonian, in the sense that, once a contact form has been chosen, the map $\mathfrak{g} \to \mathfrak{X}_{cont}(M)$ factors through $C^\infty(M)$. As noted above, for an arbitrary $f \in C^\infty(M)$ the associated Hamiltonian vector field satisfies

$$
\mathcal{L}(X_f)\theta = (\xi \cdot f)\theta. \tag{6}
$$

Since θ is G-invariant, for any $X \in \mathfrak{g}$ we have $\mathcal{L}(X_M)\theta = 0$. Since X_M is the Hamiltonian vector field associated to Φ_θ^X, we may deduce that

$$
\xi \cdot \Phi_\theta^X = 0,
$$

so that $(\Phi_\theta^X, X_M) \in \mathcal{P}_b(M, \Omega)$. Hence, we can consider the quantization of the contact manifold (M, θ) equipped with a group of symmetries G, in terms of the smaller Lie subalgebra generated by the momentum map components Φ_θ^X.

3.5 Quantum bundles

Having established contact analogues of the symplectic description of a classical system, we now consider the corresponding construction of a quantum system. We begin with *quantum bundles*, the generalization of prequantum line bundles described in [23].

Let M be a compact manifold, let $E \subset TM$ be a subbundle equipped with a symplectic form Ω, and let $\pi : \mathbb{L} \to M$ be a Hermitian line bundle equipped with metric h and connection ∇.

Definition 3.12. *We say that* $(\mathbb{L}, h, \nabla) \to (M, E, \Omega)$ *is a* **quantum bundle** *if the restriction of the curvature form of* ∇ *to* $E \otimes E$ *is equal to* Ω.

Given a compact contact manifold (M, E) and a choice of contact form θ, we have the symplectic structure given by $\Omega = -d\theta|_{E \otimes E}$ on E. Let us suppose that $(\mathbb{L}, h, \nabla)$ is a quantum bundle over (M, E, Ω). Then, following the symplectic case, we can consider the Hilbert space

$$\mathcal{H} = \Gamma_{L^2}(M, \mathbb{L}),$$

equipped with the inner product

$$< s_1, s_2 >= \int_M h(s_1, s_2) \theta \wedge \frac{d\theta^n}{n!}.$$

Using the contact momentum map $\Phi_\theta : M \to \mathfrak{g}^*$, we can again define the operators

$$A_X = -\nabla_{X_M} + i\pi^* \Phi_\theta^X, \quad X \in \mathfrak{g}$$

on $\mathcal{H}$. Since ∇ preserves the metric h, and h is G-invariant, we have that

$$0 = X_M \cdot h(s_1, s_2) = h(\nabla_{X_M} s_1, s_2) + h(s_1, \nabla_{X_M} s_2),$$

from which we see that the operators A_X are skew-Hermitian.

3.6 CR polarizations

As in the symplectic case, it is desirable to cut down the Hilbert space $\mathcal{H}$ to a smaller subspace. Since our contact manifold (M, E) is odd-dimensional, we cannot define a complex polarization on M. Instead, we make the following definition:

Definition 3.13. *A subbundle* $\mathcal{P} \subset T_\mathbb{C}M$ *will be called a* **CR polarization** *of the contact manifold* (M, E) *provided that* $\mathcal{P}$ *is formally integrable,* $\mathcal{P} \cap \overline{\mathcal{P}} = 0$, *and* $\mathcal{P} \oplus \overline{\mathcal{P}} = E \otimes \mathbb{C}$.

In other words, a CR polarization is simply a CR structure on M whose Levi distribution is the contact distribution E. Let us assume then, for the remainder of this note, that M is a strongly pseudoconvex CR manifold with CR structure $E_{1,0} \subset T_\mathbb{C}M$. Let $E \subset TM$ be the corresponding Levi distribution, and $J \in \text{End}(E)$ the fibrewise complex structure on E whose $+i$-eigenbundle is $E_{1,0}$. We choose a contact form θ such that the Webster metric

$$g_\theta(X, Y) = -d\theta(X, JY) + \theta(X)\theta(Y), \quad X, Y \in TM,$$

is Riemannian. We note that the $\mathbb{C}$-bilinear extension of g_θ to $T_\mathbb{C}M$ is Hermitian, and its restriction to $E \otimes \mathbb{C}$ agrees with the Levi form

$$L_\theta(Z, W) = -id\theta(Z, \overline{W}), \quad Z, W \in E_{1,0}.$$

Given the above data, it is well-known (see [23], for example) that there exists a unique linear connection ∇^{TW} on M, the *Tanaka-Webster connection*, such that:

(i) $\nabla^{TW}_X \Gamma(M, E) \subset \Gamma(M, E)$, for all $X \in \Gamma(M, TM)$,

(ii) $\nabla^{TW} J = \nabla^{TW} g_\theta = \nabla^{TW} \theta = 0$.

(iii) The torsion $T_{TW}(X, Y)$ of ∇^{TW} is *pure*: for any $Z, W \in E_{1,0}$ and $X \in TM$, it satisfies

$$T_{TW}(Z, W) = 0$$
$$T_{TW}(Z, \overline{W}) = 2d\theta(Z, \overline{W})\xi$$
$$T_{TW}(\xi, JX) = -JT_{TW}(\xi, X),$$

where ξ denotes the Reeb field associated to θ.

Now, the Reeb field induces a splitting $T_\mathbb{C}M = E_{1,0} \oplus E_{0,1} \oplus \mathbb{C}\xi$ of the complexified tangent bundle. Let us denote by $\widehat{T} = T_\mathbb{C}M/E_{0,1} \cong E_{1,0} \oplus \mathbb{C}\xi$. We then obtain a bigrading of the space of complexified differential forms on M, given by

$$\mathcal{A}^k_\mathbb{C}(M) = \sum_{p+q=k} \mathcal{A}^{p,q}(M),$$

where

$$\mathcal{A}^{p,q}(M) = \bigwedge^p \widehat{T}^* \otimes \bigwedge^q E^{0,1},$$

where $E^{0,1} = (E_{0,1})^*$. For $k = p + q$ we let $\pi^{p,q} : \mathcal{A}^k(M) \to \mathcal{A}^{(p,q)}(M)$ denote the natural projection. Following [24] we define the *tangential Cauchy-Riemann operator*

$$\overline{\partial}_b = \pi^{p,q+1} \circ d : \mathcal{A}^{(p,q)}(M) \to \mathcal{A}^{(p,q+1)}(M), \tag{7}$$

where $d : \mathcal{A}^r(M) \to \mathcal{A}^{r+1}(M)$ is the usual de Rham differential.

We note that for any $f \in C^\infty(M, \mathbb{C})$ and $Z \in E_{1,0}$, we have

$$(\overline{\partial}_b f)(\overline{Z}) = \overline{Z} \cdot f.$$

Definition 3.14. *We say that a function $f \in C^\infty(M, \mathbb{C})$ is **CR-holomorphic** if $\overline{\partial}_b f = 0$.*

Let us now introduce the CR analogue of a holomorphic vector bundle [23, 25]:

Definition 3.15. *Let $(M, E_{1,0})$ be a strongly pseudoconvex CR manifold. We say that a complex vector bundle $\mathcal{V} \to (M, E_{1,0})$ is **CR holomorphic** if there exists a differential operator*

$$\overline{\partial}_\mathcal{V} : \Gamma(M, \mathcal{V}) \to \Gamma(M, E^{0,1} \otimes \mathcal{V})$$

such that for any $u \in \Gamma(M, \mathcal{V})$, $f \in C^\infty(M, \mathbb{C})$ and $Z, W \in E_{1,0}$,

$$\overline{\partial}_\mathcal{V}(fu) = f(\overline{\partial}_\mathcal{V} u) + (\overline{\partial}_b f) \otimes u$$
$$[\overline{Z}, \overline{W}]u = \overline{Z}\,\overline{W}u = \overline{W}\,\overline{Z}u,$$

where $\overline{Z}u = \iota(\overline{Z})(\overline{\partial}_\mathcal{V} u)$.

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Now, suppose we are given a CR-holomorphic vector bundle $\mathcal{V} \to (M, E_{1,0})$, equipped with a Hermitian metric h. We say that a connection ∇ on $\mathcal{V}$ is *Hermitian* if $\nabla h = 0$, and $\nabla^{0,1} := \nabla|_{E_{0,1}} = \overline{\partial}_{\mathcal{V}}$. By a result of Urakawa [25], and an extension of this result in [23], such connections are uniquely determined by a trace defined with respect to $\Omega = -d\theta$.

Let us consider the trivial line bundle $\mathbb{L} = M \times \mathbb{C}$ over $(M, E_{1,0})$, with the Hermitian metric $h_x((x, z_1), (x, z_2)) = z_1 \overline{z}_2$. The operator $\overline{\partial}_{\mathbb{L}}$ defined by

$$(\overline{\partial}_{\mathbb{L}} s) = (x, (\overline{\partial}_b f)_x), \quad \text{for } s(x) = (x, f(x)), \tag{8}$$

makes $\mathbb{L}$ into a CR-holomorphic line bundle. If we equip $\mathbb{L}$ with the connection ∇ defined by

$$\nabla_X s = i(\frac{1}{if} df(X) - \theta(X)) s,$$

where $s(x) = (x, f(x))$, then the curvature form of ∇ is equal to Ω [23], making $(\mathbb{L}, h, \nabla)$ into a quantum bundle over $(M, E_{1,0}, \Omega)$. Moreover, the connection ∇ is Hermitian; we have $\nabla^{0,1} = \overline{\partial}_{\mathbb{L}}$, and the following is therefore immediate:

Proposition 3.16. *Let $(M, E_{1,0})$ be a strongly pseudoconvex CR manifold with Levi distribution E. Let $\mathcal{P}$ be the CR polarization of (M, E) given by $E_{1,0}$, and let $(\mathbb{L}, h, \nabla)$ be the quantum bundle defined above. Then the polarized sections of $\mathbb{L}$ are the CR-holomorphic sections of $\mathbb{L}$, defined by $\overline{\partial}_{\mathbb{L}} s = 0$. Thus, the space of polarized sections is isomorphic to the space of CR-holomorphic functions on M.*

Remark 3.17. This agrees with the answer obtained in [7] for the homogeneous quantization of the symplectic cone given by the symplectization of an embedded strongly pseudoconvex CR manifold.

Remark 3.18. In symplectic geometry, a polarization is given in general by a Lagrangian subbundle $\mathcal{P}$ of $TM \otimes \mathbb{C}$, with no condition on the rank of $\mathcal{P} \cap \overline{\mathcal{P}}$. (When this rank is maximal, the polarization is called *totally real*.) It would be interesting to consider other versions of polarization in the contact setting; in general, the most natural definition of a contact polarization would be that of a *Legendrian* subbundle of the complexified tangent bundle.

4 Dirac operators and index theory

We will now review the theory of Clifford bundles and Dirac operators in the setting of Kähler manifolds, for which the main reference is [26]. Following that, we will develop an analogous theory for strongly pseudoconvex CR manifolds.

4.1 The Kähler case

Given an even-dimensional Riemannian manifold M with metric g, we can form the *Clifford bundle* $\mathbb{C}l(M) \to M$ whose fibre over $x \in M$ is the complexified Clifford algebra of $T_x^* M$ with respect to the metric g. The bundle $\mathbb{C}l(M)$ is equipped with the connection ∇^{LC} induced by the Levi-Civita connection.

Definition 4.1. *A $\mathbb{Z}_2$-graded vector bundle $\mathcal{V} \to M$ is called a **Clifford module** if there exists a homomorphism of graded algebras $a \in \mathbb{Cl}(M) \mapsto \mathbf{c}(a) \in \mathrm{End}(\mathcal{V})$. We call a $\mathbb{Z}_2$-graded vector bundle $\mathcal{S} \to M$ a **spinor bundle** if $\mathcal{S}$ is a Clifford module, and $\mathbb{Cl}(M) \to \mathrm{End}(\mathcal{S})$ is an isomorphism.*

Remark 4.2. Given any vector bundle $\mathcal{W} \to M$ and a Clifford module $\mathcal{V} \to M$, the tensor product bundle $\mathcal{V} \otimes \mathcal{W}$ is again a Clifford module, with respect to the Clifford action $\mathbf{c}(a) \otimes 1$. If M is equipped with a spin structure, then there is a canonical spinor bundle $\mathcal{S}$, and any other Clifford module is of the form $\mathcal{S} \otimes \mathcal{W}$ for some vector bundle $\mathcal{W}$.

Definition 4.3. *We say that a connection $\nabla^{\mathcal{V}}$ on a Clifford module $\mathcal{V}$ is a **Clifford connection** if for every $a \in \Gamma(M, \mathbb{Cl}(M))$ and $X \in \Gamma(M, TM)$, we have*

$$[\nabla_X^{\mathcal{V}}, \mathbf{c}(a)] = \mathbf{c}(\nabla_X^{LC} a).$$

Given such a connection, we define a Dirac operator $\slashed{D} : \Gamma(M, \mathcal{V}^+) \to \Gamma(M, \mathcal{V}^-)$ given by $\slashed{D} = \mathbf{c} \circ \nabla^{\mathcal{V}}$.

Remark 4.4. Let $\mathcal{W}$ be a vector bundle with connection $\nabla^{\mathcal{W}}$, and let $\mathcal{S}$ be a spinor bundle equipped with a Clifford connection $\nabla^{\mathcal{S}}$. The tensor product connection, given for $s \in \Gamma(M, \mathcal{S})$ and $w \in \Gamma(M, \mathcal{W})$ by

$$\nabla(s \otimes w) = \nabla^{\mathcal{S}} s \otimes w + s \otimes \nabla^{\mathcal{W}} w \tag{9}$$

is then a Clifford connection on $\mathcal{S} \otimes \mathcal{W}$.

Let us now assume that M is a Kähler manifold. Let h be a Hermitian metric on $T_{\mathbb{C}} M$ with underlying Kähler metric g, and let $J \in \mathrm{End}(TM)$ be the complex structure. Since M is Kähler if and only if $\nabla^{LC} J = 0$, the Levi-Civita connection preserves the bundles $T^{1,0} M$ and $T^{0,1} M$. We have the spinor bundle

$$\mathcal{S} = \bigwedge (T^{0,1} M)^*,$$

with Clifford action given by

$$\mathbf{c}(\alpha)\nu = \sqrt{2}(\varepsilon(\alpha^{0,1}) - \iota(\alpha^{1,0}))\nu, \tag{10}$$

where we have identified $(T^{1,0})M^*$ with $T^{0,1} M$ by means of the Hermitian metric. The connection induced by the Levi-Civita connection is then a Clifford connection for $\mathcal{S}$.

Given a Hermitian vector bundle $\mathcal{W} \to M$ with metric $h_{\mathcal{W}}$ and connection $\nabla^{\mathcal{W}}$, we have the decomposition $\nabla^{\mathcal{W}} = \nabla^{1,0} \oplus \nabla^{0,1}$ given by the restrictions of $\nabla^{\mathcal{W}}$ to $T^{1,0} M$ and $T^{0,1} M$, respectively. The bundle $\mathcal{W}$ is holomorphic if and only if there exists a unique differential operator

$$\overline{\partial}_{\mathcal{W}} : \mathcal{A}^{p,q}(M, \mathcal{W}) \to \mathcal{A}^{p,q+1}(M, \mathcal{W})$$

such that in any local holomorphic chart, $\overline{\partial}_{\mathcal{W}} = \sum \varepsilon(d\overline{z}^i) \dfrac{\partial}{\partial \overline{z}^i}$. If $\mathcal{W}$ is holomorphic, with Hermitian metric $h_{\mathcal{W}}$, then we have [26, Proposition 3.65]:

Theorem 4.5. *There exists a unique connection $\nabla^{\mathcal{W}}$ (the canonical connection) on $\mathcal{W}$ such that*

(i) $\nabla^{\mathcal{W}} h_{\mathcal{W}} = 0.$

(ii) $\nabla^{0,1} = \overline{\partial}_{\mathcal{W}}.$

Given a holomorphic vector bundle $\mathcal{W}$, the tensor product bundle $\mathcal{S} \otimes \mathcal{W}$ is a Clifford module, with the Clifford connection ∇ given by (9), and we have the following [26, Proposition 3.67]:

Theorem 4.6. *The Dirac operator associated to the Clifford connection ∇ on $\mathcal{S} \otimes \mathcal{W}$ is given by*

$$\slashed{D} = \sqrt{2}\left(\overline{\partial}_{\mathcal{W}} + \overline{\partial}_{\mathcal{W}}^*\right). \tag{11}$$

Now, the operator $\overline{\partial}_{\mathcal{W}}$ satisfies $\overline{\partial}_{\mathcal{W}} \circ \overline{\partial}_{\mathcal{W}} = 0$ on the complex

$$0 \to \mathcal{A}^{0,0}(M, \mathcal{W}) \to \mathcal{A}^{0,1}(M, \mathcal{W}) \to \mathcal{A}^{0,2}(M, \mathcal{W}) \to \cdots$$

allowing us to define the Dolbeault cohomology groups $H^{0,q}(M, \mathcal{W})$, which in turn are isomorphic to the sheaf cohomology groups $H^q(M, \mathcal{O}(\mathcal{W}))$, where $\mathcal{O}(\mathcal{W})$ denotes the sheaf of holomorphic sections of $\mathcal{W}$.

Let us denote by $[\text{index}(\slashed{D})]$ the $\mathbb{Z}_2$-graded vector space

$$[\text{index}(\slashed{D})] = \ker \slashed{D}^+ \oplus \ker \slashed{D}^-,$$

where $\slashed{D}^+ = \slashed{D}|_{\mathcal{A}^{0,2\bullet}(M,\mathcal{W})}$ is assigned even grading, and $\slashed{D}^- = \slashed{D}|_{\mathcal{A}^{0,2\bullet+1}(M,\mathcal{W})} = (\slashed{D}^+)^*$ is given odd grading.

By the usual Hodge theory for the Dolbeault complex, we obtain the equality of $\mathbb{Z}_2$-graded vector spaces

$$[\text{index}(\slashed{D})] = \sum(-1)^i H^{0,i}(M, \mathcal{W}) \cong \sum(-1)^i H^i(M, \mathcal{O}(\mathcal{W})).$$

Thus, for the case $\mathcal{W} = \mathbb{L}$ we have recovered our quantization space $Q(M)$ as the index of the Dirac operator $\slashed{D} = \sqrt{2}\left(\overline{\partial}_{\mathbb{L}} + \overline{\partial}_{\mathbb{L}}^*\right)$, and we can compute its dimension using the Riemann-Roch formula:

$$\text{index}(\slashed{D}) = \frac{1}{(2\pi i)^n} \int_M \text{Td}(TM) \, \text{Ch}(\mathbb{L}).$$

Moreover, when M is equipped with a Hamiltonian G-action then $Q(M)$ becomes a virtual G-representation, then the associated virtual character is given near the identity, for $X \in \mathfrak{g}$ sufficiently small, by the equivariant Riemann-Roch formula:

$$\chi(e^X) = \text{index}^G(\slashed{D})(e^X) = \frac{1}{(2\pi i)^n} \int_M \text{Td}(TM, X) \, \text{Ch}(\mathbb{L}, X),$$

with similar formulas near other elements $g \in G$ (see [26]).

4.2   The Contact Case

Let us now assume that M is a strongly pseudoconvex CR manifold, with CR structure $E_{1,0} \subset T_\mathbb{C} M$ and contact form θ. We let $\mathbb{Cl}(E)$ denote the bundle of Clifford algebras over M whose fibre over $x \in M$ is the complexified Clifford algebra of E_x^* with respect to Euclidean form

$$G_\theta(X, Y) = \Omega(X, JY), \quad X, Y \in E = \ker \theta,$$

where $\Omega = -d\theta$. The CR structure is not preserved by the Levi-Civita connection, but given the choice of contact form θ, M is equipped with a canonical connection; namely, the Tanaka-Webster connection. Since $\nabla^{TW} J = 0$ and $\nabla_X^{TW} Y \in E$ for all $X \in \Gamma(M, TM)$ and all $Y \in \Gamma(M, E)$, the Tanaka-Webster connection induces a connection on $\mathbb{Cl}(E)$. Moreover, since ∇^{TW} preserves the Webster metric g_θ, it is compatible with the Clifford multiplication in $\mathbb{Cl}(E)$.

As in the even-dimensional case, we will call a $\mathbb{Z}_2$-graded vector bundle $\mathcal{V} \to M$ a *Clifford module* if there exists a homomorphism of graded algebras $\mathbb{Cl}(E) \to \mathrm{End}(\mathcal{V})$, and we will call a $\mathbb{Z}_2$-graded vector bundle $\mathcal{S} \to M$ a *spinor bundle* if $\mathbb{Cl}(E) \to \mathrm{End}(\mathcal{V})$ is an isomorphism of graded algebras.

A spinor bundle for $\mathbb{Cl}(E)$ is given by the CR structure, according to

$$\mathcal{S} = \bigwedge E^{0,1}.$$

We again define the Clifford action of $\mathbb{Cl}(E)$ on $\mathcal{S}$ by (10), keeping in mind that we must take $\alpha \in E^*$ and not $\alpha \in T^*M$. (Note that the splitting of T^*M determined by the contact form θ allows us to identify E^* with a subbundle of T^*M.) Since we are now working with the Tanaka-Webster connection rather than the Levi-Civita connection, we make the following definition:

Definition 4.7. *We say that a connection $\nabla^\mathcal{V}$ on a Clifford module $\mathcal{V}$ is a* **CR-Clifford connection** *if for every $a \in \Gamma(M, \mathbb{Cl}(E))$ and $X \in \Gamma(M, TM)$, we have*

$$[\nabla_X^\mathcal{V}, \mathbf{c}(a)] = \mathbf{c}(\nabla_X^{TW} a). \tag{12}$$

Proposition 4.8. *The connection ∇ induced by the Tanaka-Webster connection on $\mathcal{S} = \bigwedge E^{0,1}$ is a CR-Clifford connection.*

Proof. Since the Tanaka-Webster connection is compatible with the Clifford multiplication, it suffices to check that (12) holds for a 1-form $\alpha \in \mathcal{A}^1(M)$. For any $s \in \Gamma(M, \mathcal{S})$, we have

$$\begin{aligned}
[\nabla_X, \mathbf{c}(\alpha)]s &= (\nabla_X \alpha^{0,1}) \wedge s + \alpha^{0,1} \wedge \nabla_X s \\
&\quad - (\iota(\nabla_X \alpha^{1,0})s - \iota(\alpha^{1,0})\nabla_X s) \\
&\quad\quad - (\alpha^{0,1} \wedge \nabla_X s - \iota(\alpha^{1,0})\nabla_X s) \\
&= (\nabla_X \alpha^{0,1}) \wedge s - \iota(\nabla_X \alpha^{1,0})s \\
&= \mathbf{c}(\nabla_X \alpha)s.
\end{aligned}$$

$\square$

Given a CR-Clifford connection on a Clifford module $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$, we can define a Dirac-like operator $\slashed{D}_b : \Gamma(M, \mathcal{V}^+) \to \Gamma(M, \mathcal{V}^-)$ by the composition

$$\slashed{D}_b : \Gamma(M, \mathcal{V}^+) \xrightarrow{\nabla^{\mathcal{V}}} \Gamma(M, T^*M \otimes \mathcal{V}^+) \xrightarrow{q} \Gamma(M, E^* \otimes \mathcal{V}^+) \xrightarrow{\mathbf{c}} \Gamma(M, \mathcal{V}^-), \qquad (13)$$

where $q : T^*M \to E^*$ is the projection of the exact sequence

$$0 \to E^0 \hookrightarrow T^*M \xrightarrow{q} E^* \to 0.$$

Let us now consider the trivial bundle $\mathbb{L} = M \times \mathbb{C} \to M$, which is a CR-holomorphic vector bundle, equipped with the operator $\overline{\partial}_{\mathbb{L}}$ given by (8). Making the identification $\Gamma(M, \mathbb{L}) \cong C^\infty(M, \mathbb{C})$, we extend $\overline{\partial}_{\mathbb{L}}$ to an operator $\overline{\partial}_{\mathbb{L}} : \mathcal{A}^{0,k}(M) \to \mathcal{A}^{0,k+1}(M)$ given by the tangential $\overline{\partial}_b$ operator. We equip $\mathbb{L}$ with the Hermitian metric h and connection ∇ given in Section 3.6; we recall that for all $Z \in \Gamma(M, E_{1,0})$ and $s \in \Gamma(M, \mathbb{L})$, we have $\nabla_{\overline{Z}} s = \overline{Z} s := \iota(\overline{Z}) \overline{\partial}_{\mathbb{L}} s$.

By [23], the $\overline{\partial}_b$ operator can be written in terms of a local frame $\{Z_i\}$ for $E_{1,0}$ with corresponding coframe $\{\theta^i\}$ according to

$$\overline{\partial}_b \alpha = \sum \overline{\theta}^i \wedge (\nabla^{TW}_{\overline{Z}_i} \alpha),$$

for any $\alpha \in \mathcal{A}^{0,k}(M)$. In other words, as an operator on $\mathcal{A}^{0,\bullet}(M)$, $\overline{\partial}_b$ is given by the composition

$$\mathcal{A}^{0,k}(M) \xrightarrow{\nabla^{TW}} \Gamma(M, T^*M \otimes \Lambda^k E^{0,1}) \xrightarrow{q} \Gamma(M, E^* \otimes \Lambda^k E^{0,1}) \xrightarrow{\varepsilon} \mathcal{A}^{0,k+1}(M),$$

where $q : T^*M \to E^*$ is the projection with respect to the splitting determined by θ, and for any $\alpha \in \Gamma(M, E^*)$ and $\gamma \in \mathcal{A}^{0,k}(M)$, $\varepsilon(\alpha) \cdot \gamma = \alpha^{0,1} \wedge \gamma$.

We note that some care must be taken in using the above decomposition of the $\overline{\partial}_b$ operator: since ∇^{TW} has torsion, we cannot write the full exterior differential d in terms of the Tanaka-Webster connection. To obtain the above decomposition, we rely on the fact that the torsion of ∇^{TW} is pure, and hence vanishes on $E_{0,1} \otimes E_{0,1}$.

Similarly, the formal adjoint of $\overline{\partial}_b$ is given locally by the expression $\overline{\partial}_b^* \gamma = -\sum \iota(\overline{Z}_i)(\nabla_{Z_i} \gamma)$. Globally, we write this as the composition

$$\mathcal{A}^{0,k}(M) \xrightarrow{\nabla^{TW}} \Gamma(M, T^*M \otimes \Lambda^k E^{0,1}) \xrightarrow{q} \Gamma(M, E^* \otimes \Lambda^k E^{0,1}) \xrightarrow{-\iota} \mathcal{A}^{k-1}(M),$$

where $\iota(\alpha) \cdot \gamma = \iota(\alpha^{1,0}) \gamma$. Here $\alpha^{1,0} \in E^{1,0}$, and we identify $E^{1,0} = \overline{E^{0,1}} \cong (E^{0,1})^* = E_{0,1}$ using the Hermitian metric determined by g_θ. Thus, we obtain the following:

Proposition 4.9. *The Dirac operator $\slashed{D}_b$ associated to the extension of the Tanaka-Webster connection ∇^{TW} to $\mathcal{S} = \bigwedge E^{0,1} \cong \mathcal{S} \otimes \mathbb{L}$, is given by*

$$\slashed{D}_b = \sqrt{2} \left(\overline{\partial}_{\mathbb{L}} + \overline{\partial}_{\mathbb{L}}^* \right).$$

We can thus interpret the index of $\slashed{D}_b$ in terms of the Kohn-Rossi cohomology groups [12]

$$H^{(0,i)}(M, E_{1,0}) = \frac{\ker(\overline{\partial}_b : \mathcal{A}^{(0,i)}(M) \to \mathcal{A}^{(0,i+1)}(M))}{\operatorname{im}(\overline{\partial}_b : \mathcal{A}^{(0,i-1)}(M) \to \mathcal{A}^{(0,i)}(M))}. \qquad (14)$$

Proposition 4.10. *As $\mathbb{Z}_2$-graded Hilbert spaces, we have an isomorphism*

$$[\text{index}(\slashed{D}_b)] \cong \sum (-1)^i H^{(0,i)}(M, E_{1,0}).$$

Proof. From [27], we have that $H^{(0,i)}(M, E_{1,0}) \cong \ker(\square_b^i)$, where $\square_b^i$ is the Kohn-Rossi Laplacian acting on $(0,i)$-forms. Since $\ker(\square_b) = \ker(\overline{\partial}_b) \cap \ker(\overline{\partial}_b^*)$ [27] and $\square_b = \slashed{D}_b^2$, we have $\ker(\square_b) = \ker(\slashed{D}_b)$, and the result follows. $\square$

Moreover, we note $H^{(0,0)}(M, E_{1,0})$ is equal to the space of CR-holomorphic functions on M, which was our candidate for the geometric quantization of (M, E). Thus, following the symplectic case, we generalize, and make the definition

$$Q(M) = [\text{index}(\slashed{D}_b)].$$

Now, $Q(M)$ is infinite-dimensional (unlike in the Kähler case), which is unsurprising given that $\slashed{D}_b$ is not an elliptic operator. However, as shown in [9], when a compact Lie group G acts on M such that the orbits of G are transverse to the contact distribution E, $\slashed{D}_b$ becomes a transversally elliptic operator. Thus, by a result of Atiyah and Singer [13], the equivariant index of $\slashed{D}_b$ is well-defined as a generalized function on G. Using the contact form θ we can define an equivariant differential form with generalized coefficients $\mathcal{J}(E, X)$ (that is, a differential form depending distributionally on $X \in \mathfrak{g}$) given by

$$\mathcal{J}(E, X) = \theta \wedge \delta_0(D\theta(X)), \tag{15}$$

where $\delta_0(x)$ denotes the Dirac delta distribution on $\mathbb{R}$, and $D\theta(X) = d\theta - \Phi_\theta^X$ is the equivariant differential of θ. This form was introduced in [9], and depends only on the contact distribution E and the group action. In particular, it is independent of the choice of contact form θ. By manipulating the equivariant index formulas of Berline-Paradan-Vergne [28, 29, 30], we showed that the equivariant index of the G-transversally elliptic operator $\slashed{D}_b$ is then given near $1 \in G$ by

$$\text{index}^G(\slashed{D}_b)(e^X) = \frac{1}{(2\pi i)^n} \int_M \text{Td}(E, X)\mathcal{J}(E, X),$$

for $X \in \mathfrak{g}$ sufficiently small, with similar formulas near other elements of G. We note that a major advantage of taking $Q(M) = [\text{index}(\slashed{D}_b)]$ as our definition of the quantization of M is that as a G-representation, it does not depend on a choice of contact form θ, whereas the geometric construction involves a choice of contact form throughout.

Remark 4.11. As noted above, the connection induced on the spinor bundle $\mathcal{S} = \bigwedge E^{0,1}$ by the Tanaka-Webster connection is a CR-Clifford connection. Given any other vector bundle $\mathcal{W} \to M$ with connection $\nabla^{\mathcal{W}}$, the tensor product connection given by (9), where $\nabla^{\mathcal{S}}$ is the connection induced by ∇^{TW}, is a Clifford connection on $\mathcal{S} \otimes \mathcal{W}$: for any section $s \otimes w \in \Gamma(M, \mathcal{S} \otimes \mathcal{W})$, we have

$$\begin{aligned}
[\nabla_X, \mathbf{c}(a) \otimes 1]s \otimes w &= \nabla_X(\mathbf{c}(a)s \otimes w) - \mathbf{c}(a) \otimes 1(\nabla_X^{\mathcal{S}}s \otimes w + s \otimes \nabla_X^{\mathcal{W}}w) \\
&= ([\nabla_X^{TW}, \mathbf{c}(a)]s) \otimes w \\
&= (\mathbf{c}(\nabla_X^{TW}a)s) \otimes w \\
&= (\mathbf{c}(\nabla_X^{TW}a) \otimes 1)s \otimes w.
\end{aligned}$$

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Suppose that we are given a CR-holomorphic vector bundle $(\mathcal{W}, \overline{\partial}_\mathcal{W}) \to M$, equipped with a Hermitian metric $h_\mathcal{W}$. We can assume that $\mathcal{W}$ is equipped with a Hermitian connection $\nabla^\mathcal{W}$. The bundle $\mathcal{S} \otimes \mathcal{W}$ will then be a Clifford bundle, with CR-Clifford connection ∇ given by the tensor product connection (9). We then have the following:

Theorem 4.12. *The Dirac operator on $\mathcal{S} \otimes \mathcal{W}$ determined by the CR-Clifford connection ∇ is given by*

$$\slashed{D}_b^\mathcal{W} = \sqrt{2}\left(\overline{\partial}_\mathcal{W} + \overline{\partial}_\mathcal{W}^*\right),$$

where $\overline{\partial}_\mathcal{W}$ is the extension of the CR holomorphic operator of $\mathcal{W}$ to $\mathcal{S} \otimes \mathcal{W}$.

Proof. We need to show that the two operators agree on sections of $\mathcal{W} \otimes \bigwedge E^{0,1}$. Let $\{\overline{Z}_i\}$ be a local frame for $E_{0,1}$, with corresponding coframe $\{\overline{\theta}^i\}$ for $E^{0,1}$. We note that $\overline{\partial}_\mathcal{W}$ can be expressed locally by

$$\overline{\partial}_\mathcal{W}\alpha = \sum \overline{\theta}^i \wedge \overline{Z}_i\alpha, \tag{16}$$

where $\overline{Z}_i\alpha = \iota(\overline{Z}_i)\overline{\partial}_\mathcal{W}\alpha$. Let ∇ be the tensor product connection on $\bigwedge E^{0,1} \otimes \mathcal{W}$, and define the operator $\overline{\partial}_\nabla : \Gamma(M, \bigwedge^k E^{0,1} \otimes \mathcal{W}) \to \Gamma(M, \bigwedge^k E^{0,1} \otimes \mathcal{W})$ given by

$$\overline{\partial}_\nabla\alpha = \sum \overline{\theta}^i \wedge (\nabla_{\overline{Z}_i}\alpha).$$

It follows that $\slashed{D}_b^\mathcal{W} = \overline{\partial}_\nabla + \overline{\partial}_\nabla^*$, so it suffices to show that as operators on $\bigwedge E^{0,1} \otimes \mathcal{W}$, we have $\overline{\partial}_\nabla = \overline{\partial}_\mathcal{W}$. Let $\alpha \otimes w \in \Gamma(M, \bigwedge E^{0,1} \otimes \mathcal{W})$. Then, since $\nabla^\mathcal{W}$ is a Hermitian connection, we have that for any $\overline{Z} \in E_{0,1}$, $\nabla_{\overline{Z}}^\mathcal{W} w = \iota(\overline{Z})(\overline{\partial}_\mathcal{W} w)$, and therefore,

$$\begin{aligned}
\overline{\partial}_\nabla(\alpha \otimes w) &= \sum \overline{\theta}^i \wedge \left(\nabla_{\overline{Z}_i}^{TW}\alpha \otimes w + \alpha \otimes \nabla_{\overline{Z}_i}^\mathcal{W} w\right) \\
&= \sum \left(\overline{\theta} \wedge (\nabla_{\overline{Z}_i}^{TW}\alpha) \otimes w + \overline{\theta}^i \wedge \alpha \otimes (\iota(\overline{Z}_i)\overline{\partial}_\mathcal{W} w)\right) \\
&= (\overline{\partial}_b\alpha) \otimes w + (-1)^{|\alpha|}\alpha \wedge \sum \overline{\theta}^i \otimes \overline{Z}_i w \\
&= (\overline{\partial}_b\alpha) \otimes w + (-1)^{|\alpha|}\alpha \wedge (\overline{\partial}_\mathcal{W} w) \\
&= \overline{\partial}_\mathcal{W}(\alpha \otimes w).
\end{aligned}$$

$\square$

As shown in [9], if $\mathcal{W} \to M$ is G-equivariant, and the action of G on M is transverse to E, then the operator $\slashed{D}_b^\mathcal{W}$ is again a G-transversally elliptic operator, and its equivariant index is given near $1 \in G$ by

$$\mathrm{index}^G(\slashed{D}_b)(e^X) = \frac{1}{(2\pi i)^n}\int_M \mathrm{Td}(E, X)\,\mathrm{Ch}(\mathcal{W}, X)\mathcal{J}(E, X),$$

for $X \in \mathfrak{g}$ sufficiently small.

Remark 4.13. One can also consider the sheaf cohomology spaces $H^q(M, \Omega^p)$, where Ω^p represents the sheaf of CR-holomorphic p-forms. However, unlike in the Kähler case, one does not have an isomorphism $H^q(M, \Omega^p) \simeq H^{(p,q)}(M, E_{1,0})$ in all degrees [31].

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